Energy Conserving Time Domain Finite Element Methods for Electromagnetic

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Abstract

In this poster, time-domain mixed finite element simulations for Maxwells equations in bounded three-dimensional domains are presented. The electric and magnetic fields are discretized with Nédeléc and Raviart Thomas finite elements in space. Symplectic and Backward Euler methods are employed for temporal discretization. The obtained fields are also visualized on 3D meshes. The proposed methods are accurate both in space and time up to order 4, and parallel in space. In case of symplectic time integration they are energy conserving.

1 Introduction

Let Ω be a volume in $\mathbb{R}^3$ with boundary Γ and unit outward normal $\mathbf{n}$. Let $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{x}, t)$ denote the electric and magnetic field intensities respectively, where the time variable t belongs to some interval $(0, T)$, $T > 0$. Given a current density function $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$, specifying the applied current, Maxwell's equations state that,

$$\varepsilon \mathbf{E}_t + \sigma \mathbf{E} - \nabla \times \mathbf{H} = \mathbf{J} \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0 \quad \text{in } \Omega \times (0, T). \tag{2}$$

We shall assume a perfect conducting boundary condition on Ω i.e. $\mathbf{n} \times \mathbf{E} = 0$. We consider time-independent dielectric permittivity ε, magnetic permeability μ and electric conductivity σ.

2 Weak Formulation and Spatial Discretization

The weak solution $(\mathbf{E}, \mathbf{H})$ of the system (1)-(2) satisfies

$$(\varepsilon \mathbf{E}_t, \mathbf{\Phi}) + (\sigma \mathbf{E}, \mathbf{\Phi}) - (\mathbf{H}, \nabla \times \mathbf{\Phi}) = (\mathbf{J}, \mathbf{\Phi}) \, \forall \mathbf{\Phi} \in \mathbf{H}(\mathrm{curl}\,; \Omega), \tag{3}$$

$$(\mu \mathbf{H}_t, \mathbf{\Psi}) + (\nabla \times \mathbf{E}, \mathbf{\Psi}) = 0 \quad \forall \mathbf{\Psi} \in \mathbf{H}(\mathrm{div}; \Omega). \tag{4}$$

The method uses edge finite elements as a basis for the electric field and face finite elements for the magnetic flux density. The edge elements have tangential continuity whereas the face elements have normal continuity across interfaces. This leads to the following semi-discrete matrix equations

$$\mathbf{M}_\varepsilon^{(1)} \mathbf{e}_t + \mathbf{M}_\sigma^{(1)} \mathbf{e} = \left(\mathbf{G}^{(12)}\right)^\top \mathbf{h} + \mathbf{J}^{(1)}, \tag{5}$$

$$\mathbf{M}_\mu^{(2)} \mathbf{h}_t = -\mathbf{G}^{(12)} \mathbf{e}, \tag{6}$$

where $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are the first 1-form and 2-form mass matrices respectively. The matrix $\mathbf{G}^{(12)}$ is a discrete representation of the curl operator. $\mathbf{e}$ and $\mathbf{h}$ are the vectors of electric and magnetic fields degrees of freedom. $\mathbf{J}^{(1)}$ is a discrete current source. Since the vectors $\mathbf{e}$ and $\mathbf{h}$ have different dimensions, $\mathbf{G}^{(12)}$ is rectangular.

3 Time Discretization and Validations

For brevity, a fourth order symplectic method for the system (3)-(4) is given here as an example, Compute the number of time steps **nstep** $= \frac{t_{final} - t_0}{\Delta t}$

Set the initial conditions
$\mathbf{e}_1 \leftarrow \mathbf{e}_{Initial}, \mathbf{h}_1 \leftarrow \mathbf{h}_{Initial}$
loop over time steps.
for i=1 to nstep do
begin integration method update
$\mathbf{e}_{in} \leftarrow \mathbf{e}_i, \mathbf{h}_{in} \leftarrow \mathbf{h}_i$
update the field values
$\mathbf{e}_{out} \leftarrow \mathbf{e}_{in} + \alpha_j \Delta t (\mathbf{M}_\varepsilon^{(1)})^{-1} \Big(\left(\mathbf{G}^{(12)}\right)^\top \mathbf{h}_{in} - \mathbf{M}_\sigma^{(1)} \mathbf{e}_{in} + \mathbf{J}^{(1)} \Big)$
$\mathbf{h}_{out} \leftarrow \mathbf{h}_{in} + \beta_j \Delta t (\mathbf{M}_\mu^{(2)})^{-1} (\mathbf{G}^{(12)}) \mathbf{e}_{out}$
Update field value for this time step
$\mathbf{e}_{i+1} \leftarrow \mathbf{e}_{out}, \mathbf{h}_{i+1} \leftarrow \mathbf{h}_{out}$
end for
$\mathbf{e}_{final} \leftarrow \mathbf{e}_{nstep+1}, \mathbf{h}_{final} \leftarrow \mathbf{h}_{nstep+1}$.

The values of β and α are corresponding to forth order time integration. $\beta_1 = \frac{2+2^{\frac{1}{3}}+2^{\frac{-1}{3}}}{6}, \alpha_1 = 0, \beta_2 = \frac{1-2^{\frac{1}{3}}-2^{\frac{-1}{3}}}{6}, \alpha_2 = \frac{1}{2-2^{\frac{1}{3}}}, \beta_3 = \frac{1-2^{\frac{1}{3}}-2^{\frac{-1}{3}}}{6}, \alpha_3 = \frac{1}{1-2^{\frac{2}{3}}}, \beta_4 = \frac{2+2^{\frac{1}{3}}+2^{\frac{-1}{3}}}{6}, \alpha_4 = \frac{1}{2-2^{\frac{1}{3}}}$. The CFL stability condition for symplectic integration method is $\Delta t \leq \frac{2}{\sqrt{\rho((\mathbf{M}_\varepsilon^{(1)})^{-1}(\left(\mathbf{G}^{(12)}\right)^\top \mathbf{M}_\mu^{(2)} \mathbf{G}^{(12)})}}$, where ρ denotes the spectral radius. In Table 1, we summarize the results of the absolute errors $Err(\mathbf{E}) = \|\mathbf{E}(t^n) - \mathbf{E}_h^n\|_0$ and $Err(\mathbf{H}) = \|\mathbf{H}(t^n) - \mathbf{H}_h^n\|_0$ for the fourth order symplectic integration method.

Refinement level	Electric and magnetic fields		
	$Err(\mathbf{E})$	$Err(\mathbf{B})$	***stable time step*** Δt
l=2	2.786666	1.17524e-08	0.282302ns
l=3	0.733713	2.2434e-09	0.140619ns

Table 1: Absolute Error

We have also tested the convergence for the L-stable backward Euler and for the A-stable implicit midpoint method. Krylov and HyprePCG solver are used for inverting the 1-form (curl-conforming) and 2- form (div-conforming) mass matrices in case of serial and parallel implementation. A conjugate gradient and HypreDiagScale preconditioned linear solver are also implemented in these serial and parallel simulations. Indeed, uniform mesh refinement and several time steps suggested that the methods analyzed in this poster possess good accuracy. Figs. 1 and 2 represent the of electric and magnetic fields values on Esher and beam tetrahedron meshes respectively for the backward Euler method. The instantaneous energy is the total discrete energy that is stored in electric and magnetics fields, and the energy is computed as **Energy** $= \frac{1}{2}\left(\mathbf{e}^T \mathbf{M}_\varepsilon \mathbf{e} + \mathbf{h}^T \mathbf{M}_\mu \mathbf{h}\right)$.

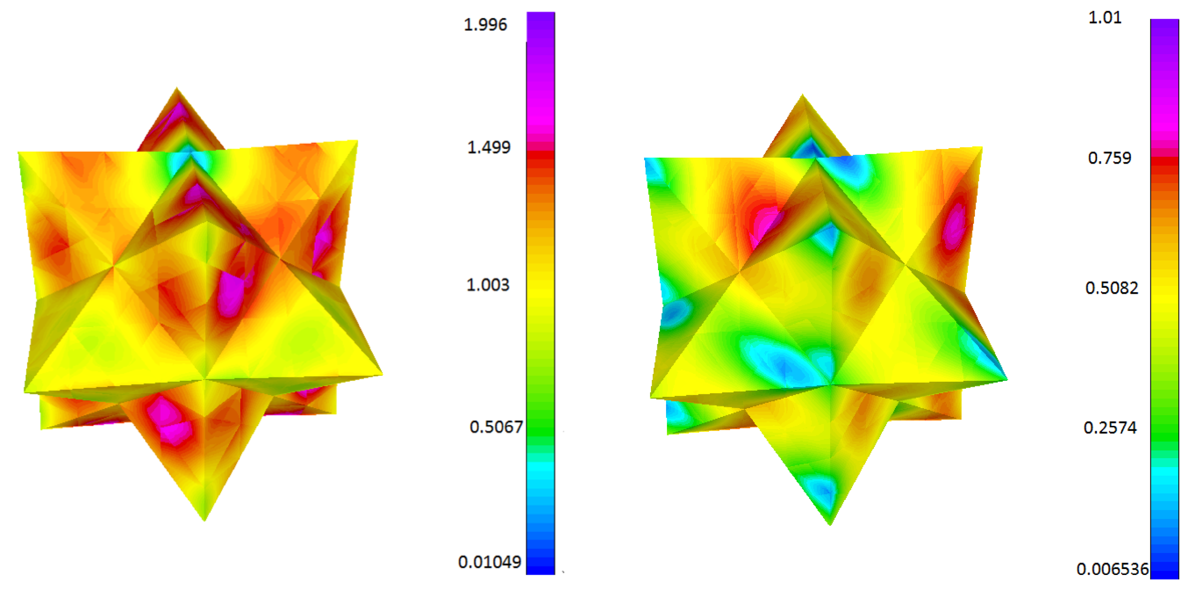

Figure 1: Electric field (left), Magnetic field (right)

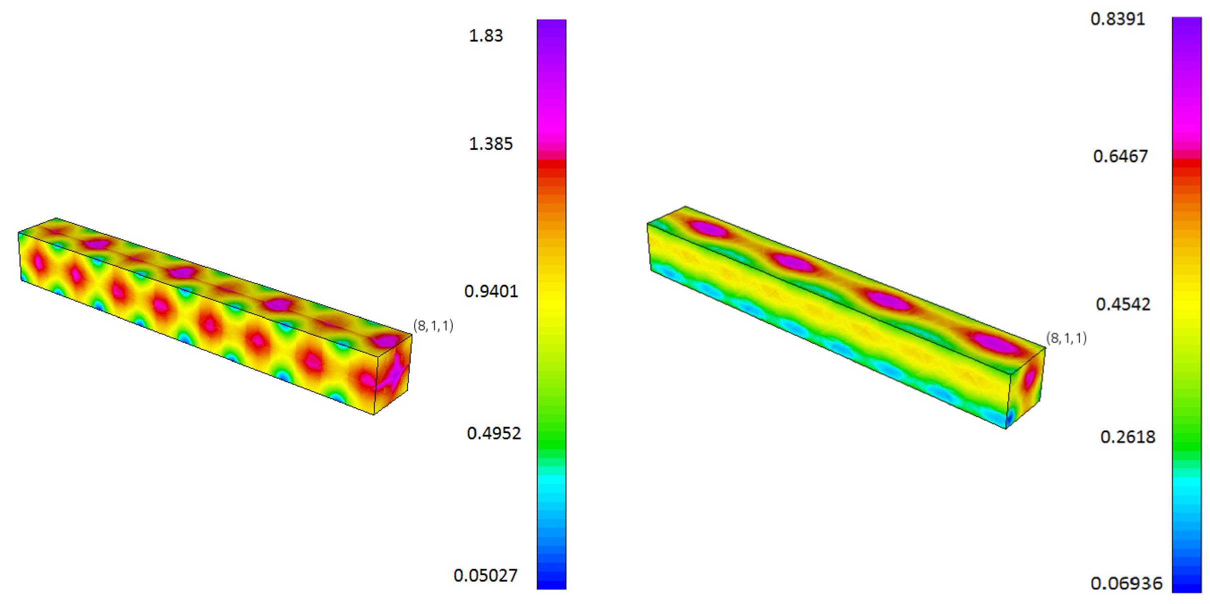

Figure 2: Electric field (left), Magnetic field (right)

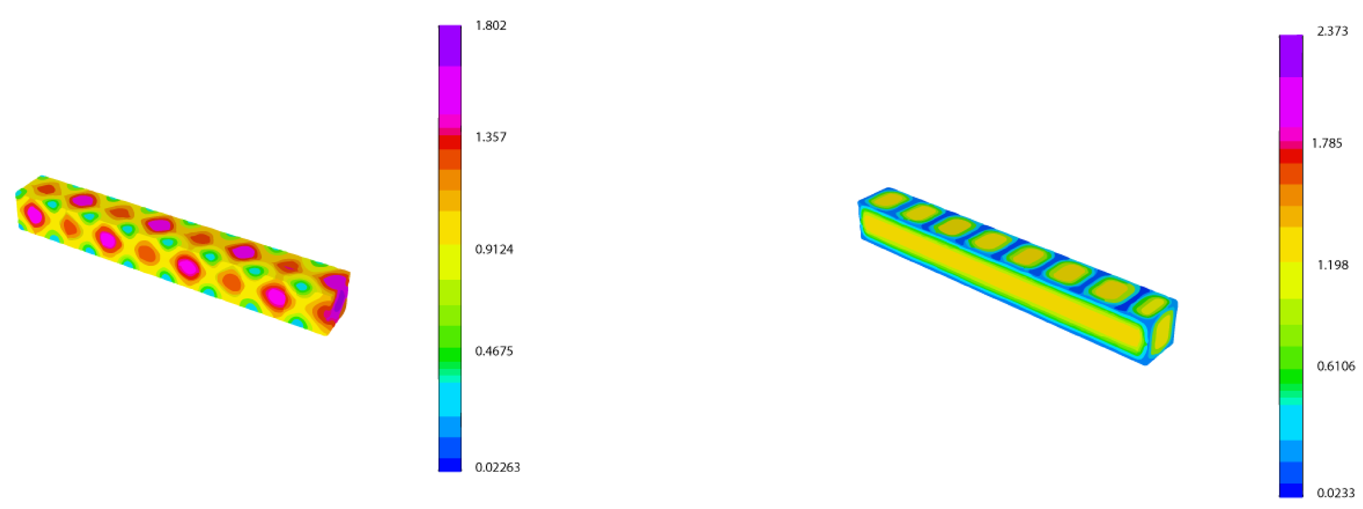

Figure 3: Discrete Energy of the system

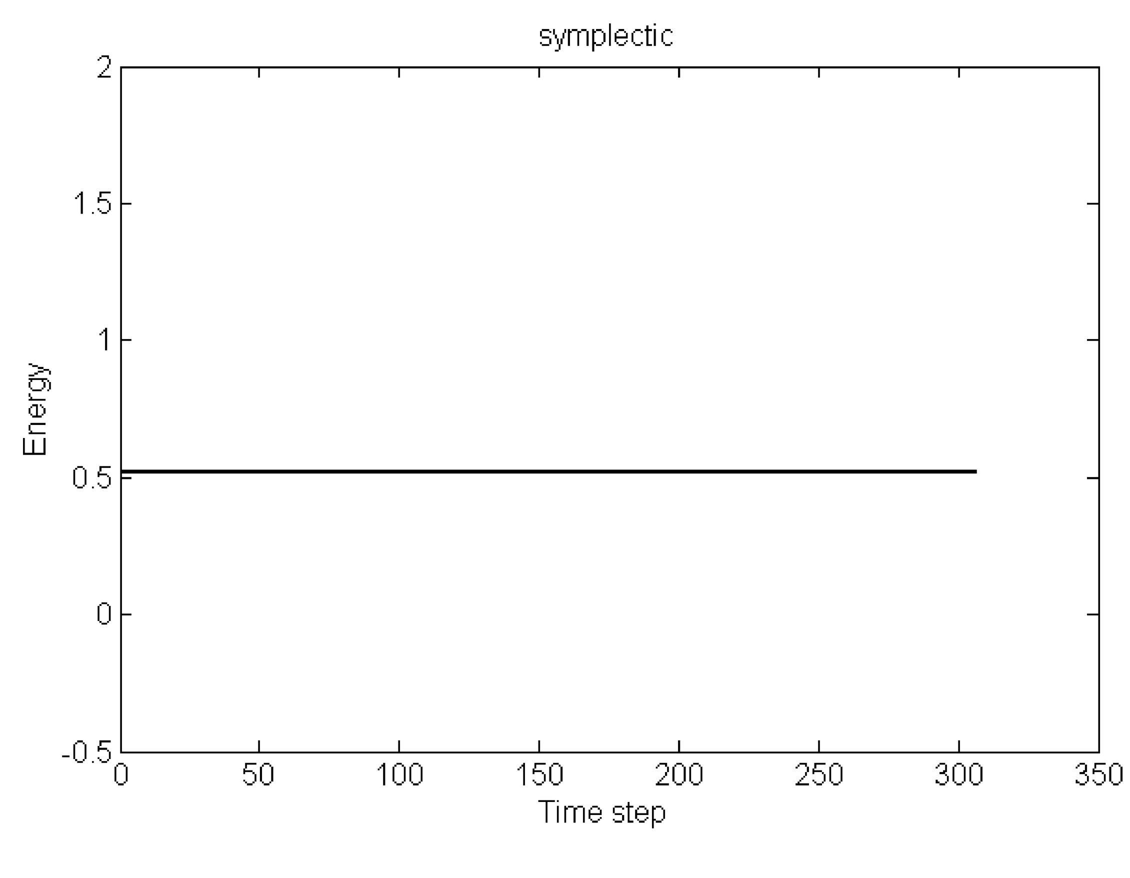

Figure 4: Discrete Energy of the system

4 Conclusion and Current Research

In this poster, Maxwell's equations simulations using finite elements of curl-conforming and div-conforming families have been presented. The simulations have been performed on 3D meshes using time-domain finite element methods. These are direct simulations of Maxwell's equations. The presented time-domain finite element methods are conditionally stable and convergent, they are accurate both in space and time up to order 4. The publications for both serial and parallel in space time domain finite element methods for linear and nonlinear Maxwell's equations are in preparation to submit in the Journal Computer Methods in Applied Mechanics and Engineering. In the ongoing work we are developing both parallel in space and time finite element methods for nonlinear models in Optics and electromagnetic in time domain. These parallel implementations are doing in MFEM, HYPRE and Xbraid.

References

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