Solving the Navier–Stokes Equations Efficiently

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July 9, 2018 / IHPCSS 2018, Ostrava

The Incompressible Navier–Stokes Equations

$$\begin{aligned}
-\nu \nabla^2 \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p &= \vec{f} && \text{in } \Omega,\\
\nabla \cdot \vec{u} &= 0 && \text{in } \Omega,\\
\vec{u} &= \vec{g}_D && \text{on } \partial\Omega_D,\\
\nu \frac{\partial \vec{u}}{\partial \vec{n}} - \vec{n} p &= 0 && \text{on } \partial\Omega_N.
\end{aligned}$$

The Linearized (Oseen) Equations

$$\begin{aligned}
-\nu \nabla^2 \vec{u}^n + (\underbrace{\vec{u}^{n-1}}_{=\vec{w}} \cdot \nabla)\vec{u}^n + \nabla p^n &= \vec{f} && \text{in } \Omega,\\
\nabla \cdot \vec{u}^n &= 0 && \text{in } \Omega,\\
\vec{u}^n &= \vec{g}_D && \text{on } \partial\Omega_D,\\
\nu \frac{\partial \vec{u}^n}{\partial \vec{n}} - \vec{n} p^n &= 0 && \text{on } \partial\Omega_N,
\end{aligned}$$

Hybridizable Discontinuous Galerkin distinguishes element-wise unknowns (U, P) from the face-wise unknowns $(\bar{U}, \bar{P})$,

$$\begin{bmatrix} N_{uu} & N_{u\bar{u}} & B_{up} & B_{u\bar{p}} \\ N_{\bar{u}u} & N_{\bar{u}\bar{u}} & 0 & B_{\bar{u}\bar{p}} \\ C_{pu} & 0 & 0 & 0 \\ C_{\bar{p}u} & C_{\bar{p}\bar{u}} & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ \bar{U} \\ P \\ \bar{P} \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Due to hybridization, N_{uu}, B_{up} and C_{pu} are block-diagonal matrices. Eliminate U and P to get

$$\begin{bmatrix} S_{\bar{u}\bar{u}} & S_{\bar{u}\bar{p}} \\ S_{\bar{p}\bar{u}} & S_{\bar{p}\bar{p}} \end{bmatrix} \begin{bmatrix} \bar{U} \\ \bar{P} \end{bmatrix} = \begin{bmatrix} L_{\bar{u}} \\ L_{\bar{p}} \end{bmatrix}$$

$Ax = b$, $\kappa(A) = ||A^{-1}||\,||A||$, $\kappa(A) \gg 1$ ill-conditioned

Preconditioner R, $\kappa(R^{-1}A) \ll \kappa(A)$

$$\underbrace{\begin{bmatrix} N_{uu} & \widetilde{N_{u\bar{u}}} & B_{up} & B_{u\bar{p}} \\ \widetilde{N_{\bar{u}u}} & \widetilde{N_{\bar{u}\bar{u}}} & 0 & \widetilde{B_{\bar{u}\bar{p}}} \\ C_{pu} & 0 & 0 & 0 \\ C_{\bar{p}u} & \widetilde{C_{\bar{p}\bar{u}}} & 0 & 0 \end{bmatrix}}_{:=A} \begin{bmatrix} U \\ \bar{U} \\ P \\ \bar{P} \end{bmatrix} = \begin{bmatrix} \widetilde{F} \\ G \\ 0 \\ \Phi \end{bmatrix},$$

$\kappa(A)$ depends on h and $\frac{UL}{\nu}$

R optimal, $\kappa(R^{-1}A)$ parameter independent.

Perfect Preconditioner

Let's write the linear system in more compact form

$$\begin{bmatrix} \mathrm{N} & \mathrm{B} \\ \mathrm{C} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{U} \\ \mathrm{P} \end{bmatrix} = \begin{bmatrix} \mathrm{F} \\ \mathrm{L} \end{bmatrix}.$$

Turns out the preconditioner

$$\begin{bmatrix} \mathrm{N} & \mathrm{B} \\ 0 & \mathrm{C}\mathrm{N}^{-1}\mathrm{B} \end{bmatrix}$$

is the "perfect" preconditioner and iterative solvers (e.g. GMRES) converges in two iterations[1].

[1] Elman, Silvester and Wathen, "Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics"

The Pressure Convection-Diffusion Preconditioner

Replace $\mathrm{C}\mathrm{N}^{-1}\mathrm{B}$ with mass pressure solve Q^{-1}, matrix-vector multiplication with Pressure Convection-Diffusion matrix F_p and a Pressure Poisson solve A_p^{-1}.

If you are interested about preliminary results, I have a demo.