Hybridizable Discontinuous Galerkin Methods for Linear Free Surface Problems

Giselle Sosa Jones

Sander Rhebergen

Department of Applied Mathematics

UNIVERSITY OF WATERLOO

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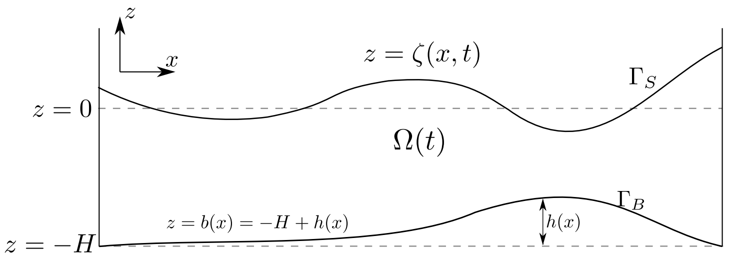
Free Surface Problem



Irrotational flow, linearized BC's (fixed domain)

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \\ \nabla\phi \cdot \mathbf{n} = \partial_t \zeta & \text{on } z = 0 \\ \partial_t \phi + \zeta = 0 & \text{on } z = 0 \\ \nabla\phi \cdot \mathbf{n} = 0 & \text{on } \Gamma_B \end{cases}$$

Discrete Problem

Discretization with HDG

Find $(\phi_h^n,\, \mathbf{q}_h^n,\, \lambda_h^n) \in W_h^p \times \mathbf{V}_h^p \times M_h^p$ s.t. for all $(w_h,\, \mathbf{v}_h,\, \mu_h) \in W_h^p \times \mathbf{V}_h^p \times M_h^p$, the following relations are satisfied

$$-(\mathbf{q}_h^n, \mathbf{v}_h)_{\mathcal{T}_h} + (\phi_h^n, \nabla \cdot \mathbf{v}_h)_{\mathcal{T}_h} - \langle \lambda_h^n, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0,$$

$$(w_h, \nabla \cdot \mathbf{q}_h^n)_{\mathcal{T}_h} + \tau \langle w_h, \phi_h^n \rangle_{\partial \mathcal{T}_h} - \tau \langle w_h, \lambda_h^n \rangle_{\partial \mathcal{T}_h} = 0,$$

$$-\langle \mathbf{q}_h^n \cdot \mathbf{n}, \mu_h \rangle_{\mathcal{E}_h} - \langle \tau \phi_h^n, \mu_h \rangle_{\mathcal{E}_h} + \langle \tau \lambda_h^n, \mu_h \rangle_{\mathcal{E}_h} + \frac{9}{4\Delta t^2} \langle \phi_h^n, \mu_h \rangle_{\Gamma_S} = g(\phi_h^{n-1}, \phi_h^{n-2})$$

Linear System

$$\begin{bmatrix} A & B^T & C^T \\ B & D & E^T \\ C & E & H \end{bmatrix} \begin{bmatrix} Q \\ \Phi \\ \Lambda \end{bmatrix} = \begin{bmatrix} R \\ F \\ L \end{bmatrix} \Rightarrow \begin{cases} \text{Step 1:} & \text{Static condensation:} \\ & \text{solve for } \Lambda \\ \text{Step 2:} & \text{Reconstruct } Q \text{ and } \Phi \end{cases}$$

Algorithm

Algorithm 1 Solving the linear free surface problem

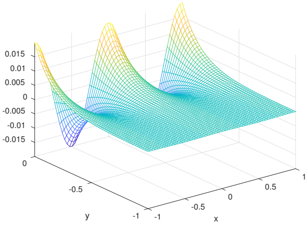
for $n = 1$ to Number of time steps **do**
    for $k = 1$ to Number of mesh elements **do**
        Assemble local matrices A_k, B_k, C_k, D_k, E_k, H_k and local RHS R_k, F_k, L_k
        Map to global matrices A, B, C, D, E, H and global RHS R, F, L
    end for
    Solve linear system for Λ
    Reconstruct Q and Φ
    Compute the wave height ζ_h^n using ϕ_h^n
    Update to the next time step
end for

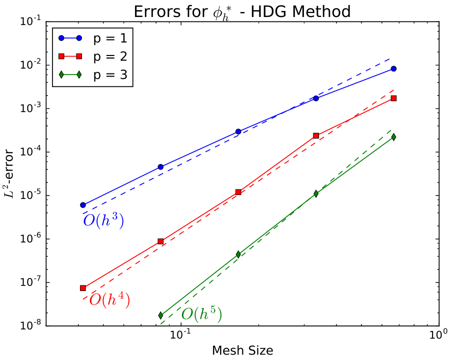
The implementation is done in the C++ library MFEM.

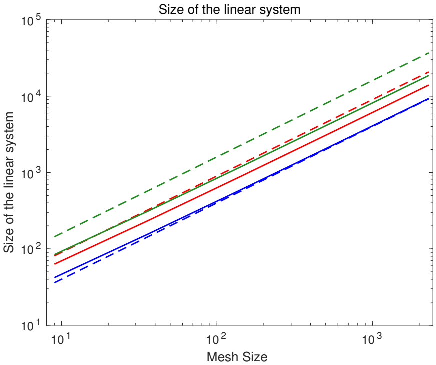
Test 1

The analytical solution is

$$\phi(x, y, t) = \phi_0 \cosh(k(y+1)) \cos(\omega t - kx)$$

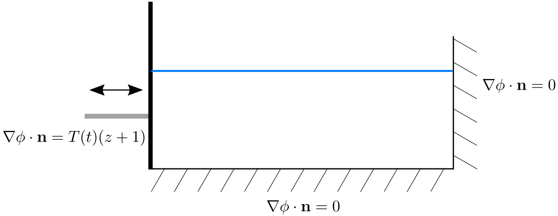




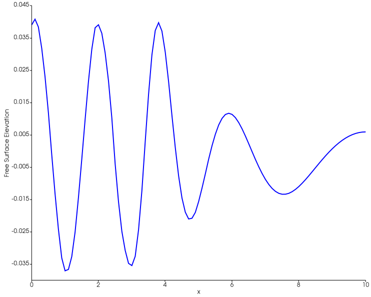


Solid line: HDG, dashed line: DG

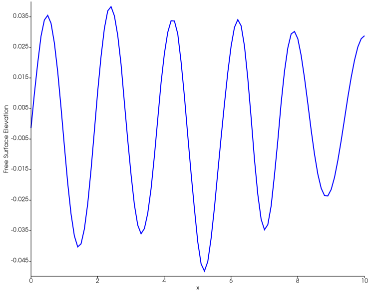
Test 2



$t = 18.4$



$t = 32.96$



I have a video of the solution!