Extracting Coherent Patterns through Spectral Properties of the Koopman Operator

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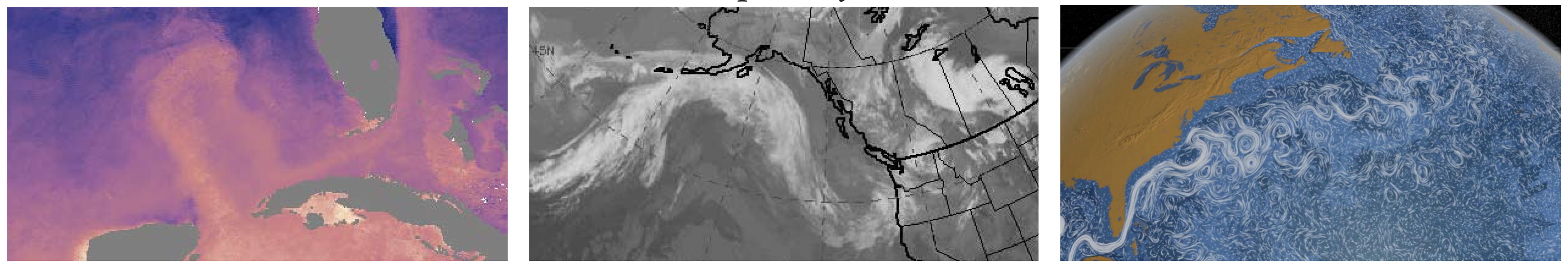
Motivation

Coherent patterns are frequently observed in oceanic currents (left and right image), planetary atmospheres (middle image), and many other natural contexts [1]. Coherent patterns are considered **Lagrangian** if they are defined from the bulk motion of a fluid parcel. Studying Lagrangian motion enables the discovery of **important transport properties** of fluid flows and extracting coherent patterns reveals the **robust material surfaces** behind complex dynamics.



Left: the Loop Current connecting the Gulf of Mexico to the Atlantic Ocean, May 1-8, 2010 (NASA). Middle: infrared satellite image of the polar jet recorded on July 4, 2000 (NOAA). Right: surface currents circulate in a high-resolution, 3D model of the Earth's oceans (NASA).

Method

Our approach for extracting coherent patterns is to study the spectra of **Koopman operators** acting on observables of the system [2].

In the setting of a time-dependent fluid flow governed by a stream function, $\Psi(t, x)$, define a space-time manifold $M = A \otimes X$, where A describes the evolution of the fluid flow and X describes the Lagrangian motions in response to that flow. $\Psi(t, x)$ can be rewritten into a **skew-product** transformation:

$$\Omega_t(a, x) = \Big(\Phi_t(a), \Psi_a(t)\Big)$$

where the dynamics are autonomous on M.

Assuming $\Phi_t(a) = a + \omega t$, the infinitesimal **generator** of the Koopman operator is:

$$W_t = \omega \frac{\partial}{\partial t} + \vec{u} \cdot \nabla.$$

For the purpose of numerical stability, we regulerize W_t by a data-driven **diffusion** maps algorithm [3]. The regulerized infintesimal generator of the Koopman operator is:

$$L = W_t - \epsilon\Delta.$$

Choose the appropriate basis for each coordinate and build the finite-dimensional matrix representation of the generator by inner product:

$$L_{i,j} = \Big\langle \phi_i, L\phi_j \Big\rangle.$$

Solve the (generalized) eigenvalue problem using spectral **Galerkin** method, $Lz_k = \lambda_k z_k$, where z_k are eigenfunctions of L.

In our approach, the **coherent patterns** are **low-Dirichlet-energy** (small roughness) **eigenfunctions** of the Koopman operators.

References

[1] George Haller. *Lagrangian Coherent Structures*. Annu. Rev. Fluid Mech. 47, 2015.

[2] Dimitrios Giannakis and Suddhasattwa Das. *Extraction and Prediction of Coherent Patterns in Incompressible Flows through Space-Time Koopman Analysis*. 2017.

[3] Dimitrios Giannakis and Andrew J. Majda. *Nonlinear Laplacian Spectral Analysis: Capturing Intermittent and Low-frequency Spatiotemporal Patterns in High-dimensional Data*. Stat. Anal. and Data Min., 2012.

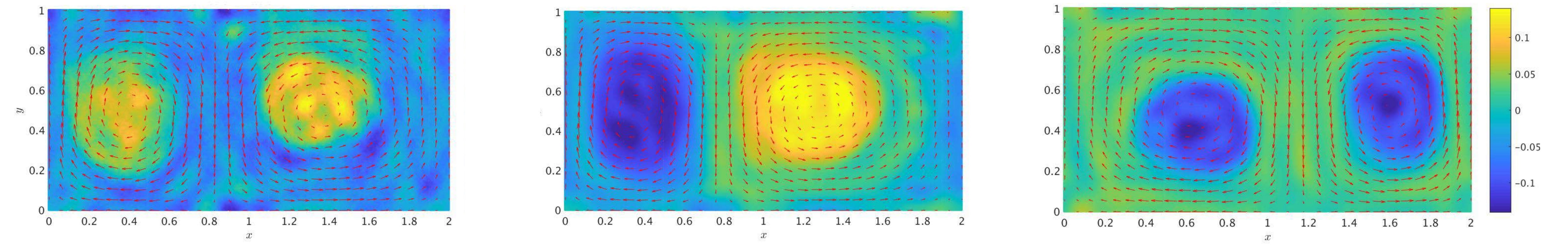
Results

A **double gyre** with **periodic** oscillation can be characterized as:

$$\Psi(x, y, t) = A\sin(\pi f(x, t))\sin(\pi y)$$

where $f(t, x) = \alpha\sin(\omega t)x^2 + (1 - 2\alpha\sin(\omega t))x$. It can be observed as a large-scale ocean circulation in the northen mid-latitude ocean basins. The parameters are chosen to be $A = 0.25$, $\alpha = 0.25$ and $\omega = \pi/\sqrt{2}$.

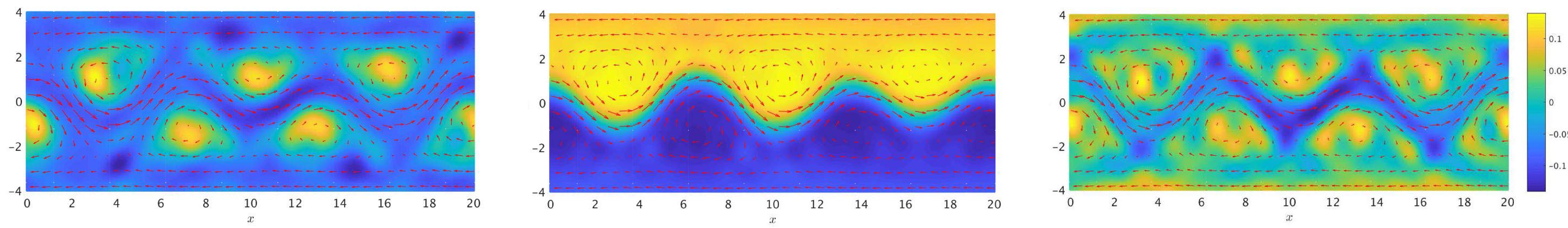
4th Eigenfunction	7th Eigenfunction	15th Eigfunction
$\lambda_4 = -4.00\text{e}{-03} - 9.870\text{e}{+00}i$	$\lambda_7 = -7.346\text{e}{-03} + 4.935\text{e}{+00}i$	$\lambda_{15} = -1.464\text{e}{-04} + 4.935\text{e}{+00}i$



The **idealized stratospheric flow** is a **quasi-periodically** perturbed form of a Bickley jet:

$$\Psi(x, y, t) = c_2 y - U_0 L\ \tanh(y/L) + A_2 U_0 L\ sech^2(y/L)\cos(k_2 x) + A_1 U_0 L\ sech^2(y/L)\cos(k_1 x - \sigma t).$$

2nd Eigenfunction	10th Eigfunction	11th Eigfunction
$\lambda_2 = 1.730\text{e}{-07} + 3.381\text{e}{-18}i$	$\lambda_{10} = 2.659\text{e}{-06} + 1.257\text{e}{-16}i$	$\lambda_{11} = -3.533\text{e}{-06} - 6.716\text{e}{-19}i$



The eigenfunctions capture two types of coherent patterns, the **stationary** and **periodically varying**. The stationary patterns correspond to the eigenvalues with imaginary parts **near zero**. The periodically varying patterns correspond to eigenvalues with **nonzero** imaginery parts.

Future Work

One of the challenges we would like to tackle in the future is the treatment of the continuous spectra of Koopman operators under **chaotic** dynamics. The geophysical convection in the actual ocean or atmosphere is significantly more chaotic than any idealized model. Coherent patterns persist in reality despite chaotic convection. With the eventual goal of capturing geophysical phenomena in a **realistic** way, we will model Lagrangian tracer advection in a Lorenze 63 driven stratospheric flow:

$$\begin{aligned}\Psi(x, y) = {} & c_4 y - U_0\tanh(y/L) \\ & + A_4 U_0 sech^2(y/L)\cos(k_4 x) \\ & + \sum_{i=1}^{3} A_i U_0 sech^2(y/L)\cos(k_i x - a_i)\end{aligned}$$

driven by Lorenze 63,

$$\begin{cases}\dot{a}_1 = \sigma(a_2 - a_1) \\ \dot{a}_2 = a_1(\rho - a_3) - a_2 \\ \dot{a}_3 = a_1 a_2 - \beta a_3.\end{cases}$$